

REPRESENTATIONS BY $x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + x_1x_3 + x_1x_4 + x_2x_4$

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ABSTRACT. Let $r_Q(n)$ be the representation number of a nonnegative integer n by the quaternary quadratic form $Q = x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + x_1x_3 + x_1x_4 + x_2x_4$. We first prove the identity $r_Q(p^2n) = r_Q(p^2)r_Q(n)/r_Q(1)$ for any prime p different from 13 and any positive integer n prime to p , which was conjectured in [2]. And, we explicitly determine a concise formula for the number $r_Q(n^2)$ as well for any integer n .

1. INTRODUCTION

Let r be a positive integer and

$$Q(x_1, \dots, x_r) = \sum_{i < j} a_{ij}x_i x_j + \frac{1}{2} \sum_i a_{ii}x_i^2$$

be the quadratic form associated with an $r \times r$ integral positive definite symmetric matrix (a_{ij}) with even diagonal entries. It is one of the important problems in number theory to find the number of solutions of the equation

$$Q(\mathbf{x}) = n \quad (\mathbf{x} \in \mathbb{Z}^r)$$

for a given nonnegative integer n . In the case of $r = 2$ it was well studied by Fermat, Lagrange, Gauss and Dirichlet, and general cases were considered systematically by Minkowski, Hasse and Siegel. Only in very special cases does exist a satisfactory formula for the representation number

$$r_Q(n) = \#\{\mathbf{x} \in \mathbb{Z}^r ; Q(\mathbf{x}) = n\}.$$

For instance, when $Q = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2$, Jacobi ([6, pp.159–170]) gave the formula

$$r_Q(n) = -4 \sum_{d>0, d|n} \chi_{-4}(d)d^2 + 16 \sum_{d>0, d|n} \chi_{-4}(n/d)d^2,$$

where $\chi_{-4}(d) = (\frac{-4}{d})$. This number $r_Q(n)$ is exactly the Fourier coefficient of the corresponding Eisenstein series ([5, §11.3]). One can also refer to [3, §31] for some concrete examples related to hypergeometric series.

Now, let us consider the quadratic form

$$Q(x_1, x_2, x_3, x_4) = x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + x_1x_3 + x_1x_4 + x_2x_4 \quad \text{associated with the matrix } \begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}.$$

If $\Theta_Q(\tau) = \sum_{n=0}^{\infty} r_Q(n)q^n$ ($q = e^{2\pi i\tau}$) is the theta function associated with Q , then $\Theta_Q(\tau)$ lies in the space $\mathcal{M}_{13}(13, \chi_{13})$ of dimension 2 which consists of all modular forms for $\Gamma_0(13)$ associated with the character $\chi_{13}(\cdot) = (\frac{13}{\cdot})$ ([9, Corollary 4.9.5]). Eum et al ([2, Example 3.4]) recently provided a basis of the space $\mathcal{M}_2(\Gamma_1(13))$ of dimension 13, which consists of modular forms for $\Gamma_1(13)$, in terms of Klein forms and expressed $\Theta_Q(\tau)$ as a linear combination of the basis elements. On the other hand, they happened to find in the process an interesting identity

$$r_Q(p^2n) = \frac{r_Q(p^2)r_Q(n)}{r_Q(1)} \quad \text{for any prime } p \text{ other than 13 and any positive integer } n \text{ prime to } p. \quad (1.1)$$

2010 *Mathematics Subject Classification.* Primary 11E25; Secondary 11F11, 11F25, 11M36.

Key words and phrases. Eisenstein series, Hecke operators, modular forms, representations by quadratic forms.

This research was partially supported by Basic Science Research Program through the NRF of Korea funded by MEST (2010-0001654). The second named author is partially supported by TJ Park Postdoctoral Fellowship.

But, they could give only a conditional proof by applying Hecke operators on $\Theta_Q(\tau)$ as follows: if p is a prime satisfying the relation

$$r_Q(p^2) = r_Q(1)(1 + \chi_{13}(p)p + p^2), \quad (1.2)$$

then (1.1) is true ([2, Proposition 4.3]). For example, each prime p ($\neq 13$) less than or equal to 347 satisfies (1.2). And so, the proof of (1.1) has remained open.

In this paper, we shall completely prove the conjecture (1.1) (Theorem 3.3) by using the fact that the space $\mathcal{M}_2(13, \chi_{13})$ is generated by two Eisenstein series (Corollary 2.7). Also, we shall obtain a general formula for $r_Q(n)$ that looks like Jacobi's formula, from which we get a concise formula for $r_Q(n^2)$ for any integer n (Remark 3.4 and Table 1).

2. MODULAR FORMS AND HECKE OPERATORS

Let k be an integer. For each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we define the *weight k slash operator* $\cdot|[\gamma]_k$ on a function $f(\tau)$ on \mathbb{H} (= the complex upper half-plane) by

$$f(\tau)|[\gamma]_k := (c\tau + d)^{-k} (f(\tau) \circ \gamma),$$

where γ acts on \mathbb{H} as a fractional linear transformation $\tau \mapsto (a\tau + b)/(c\tau + d)$. Let Γ be one of the following congruence subgroups

$$\begin{aligned} \Gamma_1(N) &:= \left\{ \alpha \in \mathrm{SL}_2(\mathbb{Z}) ; \alpha \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}, \\ \Gamma_0(N) &:= \left\{ \alpha \in \mathrm{SL}_2(\mathbb{Z}) ; \alpha \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\} \end{aligned}$$

for a positive integer N . A holomorphic function $f(\tau)$ on \mathbb{H} is called a *modular form for Γ of weight k* if

- (i) $f(\tau)|[\gamma]_k = f(\tau)$ for all $\gamma \in \Gamma$,
- (ii) $f(\tau)$ is holomorphic at every cusp ($\in \mathbb{Q} \cup \{\infty\}$) ([7, pp.125–126]). In particular, since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$ and $f(\tau) \circ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = f(\tau + 1)$ by (i), $f(\tau)$ has a Laurent series expansion with respect to

$$q := e^{2\pi i \tau}$$

of the form

$$f(\tau) = \sum_{n=0}^{\infty} a(n)q^n \quad (a(n) \in \mathbb{C}),$$

which is called the *Fourier expansion* of $f(\tau)$ (at the cusp ∞).

Moreover, if a modular form vanishes at every cusp, it is called a *cusp form*. We denote the space of all modular forms (respectively, cusp forms) for Γ of weight k by $\mathcal{M}_k(\Gamma)$ (respectively, $\mathcal{S}_k(\Gamma)$).

For a given Dirichlet character χ modulo N we define a character of $\Gamma_0(N)$ ([9, pp.79–80]), also denoted by χ , to be

$$\chi(\gamma) := \chi(d) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Let

$$\begin{aligned} \mathcal{M}_k(N, \chi) &:= \{f(\tau) \in \mathcal{M}_k(\Gamma_1(N)) ; f(\tau)|[\gamma]_k = \chi(\gamma)f(\tau) \text{ for all } \gamma \in \Gamma_0(N)\}, \\ \mathcal{S}_k(N, \chi) &:= \mathcal{S}_k(\Gamma_1(N)) \cap \mathcal{M}_k(N, \chi), \end{aligned}$$

which are subspaces of $\mathcal{M}_k(\Gamma_1(N))$ and $\mathcal{S}_k(\Gamma_1(N))$, respectively. Then we have the decomposition

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_{\chi} \mathcal{M}_k(N, \chi),$$

where χ runs over all Dirichlet characters modulo N [7, Chapter III Proposition 28]. If $\chi(-1) \neq (-1)^k$, then the space $\mathcal{M}_k(N, \chi)$ is known to be $\{0\}$ ([7, p.138]).

Proposition 2.1. *Let N be a positive integer.*

- (i) $\dim_{\mathbb{C}} \mathcal{M}_k(\Gamma_1(N)) = 0$ for any negative integer k .
- (ii) $\dim_{\mathbb{C}} \mathcal{M}_0(\Gamma_1(N)) = 1$, and hence $\dim_{\mathbb{C}} \mathcal{M}_0(N, \chi) = 0$ if χ is nontrivial.

Proof. See [9, Theorems 2.5.2 and 2.5.3]. \square

Using the Riemann-Roch Theorem, Cohen and Oesterlé ([1]) explicitly computed the following dimension formulas.

Proposition 2.2. *Let k be an integer and χ be a Dirichlet character modulo N for which $\chi(-1) = (-1)^k$. For each prime p dividing N , let r_p (respectively, s_p) denote the power of p dividing N (respectively, the conductor of χ). Define*

$$\lambda(r_p, s_p, p) := \begin{cases} p^{r'} + p^{r'-1} & \text{if } 2s_p \leq r_p = 2r' \\ 2p^{r'} & \text{if } 2s_p \leq r_p = 2r' + 1 \\ 2p^{r_p - s_p} & \text{if } 2s_p > r_p, \end{cases}$$

and

$$\nu_k := \begin{cases} 0 & \text{if } k \text{ is odd} \\ -1/4 & \text{if } k \equiv 2 \pmod{4} \\ 1/4 & \text{if } k \equiv 0 \pmod{4}, \end{cases} \quad \mu_k := \begin{cases} 0 & \text{if } k \equiv 1 \pmod{3} \\ -1/3 & \text{if } k \equiv 2 \pmod{3} \\ 1/3 & \text{if } k \equiv 0 \pmod{3}. \end{cases}$$

Then we have

$$\dim_{\mathbb{C}} \mathcal{M}_k(N, \chi) - \dim_{\mathbb{C}} \mathcal{S}_{2-k}(N, \chi) = \frac{(k-1)N}{12} \prod_{p|N} (1 + p^{-1}) + \frac{1}{2} \prod_{p|N} \lambda(r_p, s_p, p) - \nu_{2-k} \alpha(\chi) - \mu_{2-k} \beta(\chi),$$

where

$$\alpha(\chi) := \sum_{\substack{x \pmod{N} \\ x^2 + 1 \equiv 0 \pmod{N}}} \chi(x) \quad \text{and} \quad \beta(\chi) := \sum_{\substack{x \pmod{N} \\ x^2 + x + 1 \equiv 0 \pmod{N}}} \chi(x).$$

Proof. See [1, Théorème 1] or [10, Theorem 1.56]. \square

Remark 2.3. Suppose that N is a prime. Since $r_N = 1$ and $s_N = 0$ or 1 , we get $\lambda(r_N, s_N, N) = 2$. Observe that there are at most two $x \pmod{N}$ which satisfy $x^2 + 1 \equiv 0 \pmod{N}$. Furthermore, since $|\chi(x)| = 1$, we deduce $|\alpha(\chi)| \leq 2$. In a similar way, we have $|\beta(\chi)| \leq 2$. Hence

$$\dim_{\mathbb{C}} \mathcal{M}_k(N, \chi) - \dim_{\mathbb{C}} \mathcal{S}_{2-k}(N, \chi) \geq \frac{(k-1)N}{12} \cdot (1 + N^{-1}) + \frac{1}{2} \cdot 2 - \frac{1}{4} \cdot 2 - \frac{1}{3} \cdot 2 = \frac{(k-1)(N+1)}{12} - \frac{1}{6}.$$

For a nonzero integer N with $N \equiv 0$ or $1 \pmod{4}$ we denote by χ_N the Dirichlet character modulo $|N|$ defined by

$$\chi_N(d) := \text{the Kronecker symbol } \left(\frac{N}{d} \right) \quad \text{for } d \in (\mathbb{Z}/|N|\mathbb{Z})^\times.$$

Note that

$$\left(\frac{N}{-1} \right) := \begin{cases} 1 & \text{if } N > 0 \\ -1 & \text{if } N < 0. \end{cases} \quad (2.1)$$

In particular, let N be the discriminant of a quadratic field, namely, for a square-free integer m ($\neq 1$)

$$N = \begin{cases} m & \text{if } m \equiv 1 \pmod{4} \\ 4m & \text{if } m \not\equiv 1 \pmod{4}. \end{cases}$$

Then χ_N becomes a primitive Dirichlet character modulo $|N|$ ([9, pp.82–84]).

Corollary 2.4. *Let k (≥ 2) be an integer and N be a prime such that $(-1)^k N$ is the discriminant of a quadratic field. Then, $\dim_{\mathbb{C}} \mathcal{M}_k(N, \chi_{(-1)^k N}) = 2$ if and only if $(k, N) \in \{(2, 5), (2, 13), (2, 17), (3, 3), (4, 5), (5, 3)\}$.*

Proof. Since $(-1)^k N$ is the discriminant of a quadratic field and N is a prime, $(-1)^k N \equiv 1 \pmod{4}$ and $\chi_{(-1)^k N}$ is a primitive Dirichlet character modulo N . Suppose that $\dim_{\mathbb{C}} \mathcal{M}_k(N, \chi_{(-1)^k N}) = 2$. We then see that

$$\begin{aligned} 2 &= \dim_{\mathbb{C}} \mathcal{M}_k(N, \chi_{(-1)^k N}) - \dim_{\mathbb{C}} \mathcal{S}_{2-k}(N, \chi_{(-1)^k N}), \text{ because } \dim_{\mathbb{C}} \mathcal{S}_{2-k}(N, \chi_{(-1)^k N}) = 0 \text{ by Proposition 2.1,} \\ &\geq \frac{(k-1)(N+1)}{12} - \frac{1}{6} \text{ by Remark 2.3.} \end{aligned}$$

It follows that $(k-1)(N+1) \leq 26$, and so the possible pairs of (k, N) are

$$(2, 5), (2, 13), (2, 17), (3, 3), (3, 7), (3, 11), (4, 5), (5, 3), (7, 3).$$

Now, one can easily verify that $\dim_{\mathbb{C}} \mathcal{M}_k(N, \chi_{(-1)^k N}) = 2$ except for $(3, 7), (3, 11), (7, 3)$ by Propositions 2.1 and 2.2. \square

Let χ be a nontrivial primitive Dirichlet character modulo N . The *Dirichlet L-function* $L(s, \chi)$ on $s \in \mathbb{C}$ is defined by

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where we set $\chi(n) = 0$ if $\gcd(n, N) \neq 1$. As is well-known, the function converges for $\operatorname{Re}(s) > 1$. Moreover, it extends to an entire function and satisfies the following functional equation

$$L(s, \chi) = L(1-s, \bar{\chi}) \left(\frac{2\pi}{N} \right)^s \frac{S(\chi)}{\Gamma(s)} \left(\frac{e^{\pi i s/2} - \chi(-1)e^{-\pi i s/2}}{e^{\pi i s} - e^{-\pi i s}} \right),$$

where

$$S(\chi) := \sum_{a=1}^{N-1} \chi(a) e^{2\pi i a/N} \quad \text{and} \quad \Gamma(s) := \int_0^{\infty} e^{-t} t^{s-1} dt$$

([8, Chapter XIV Theorem 2.2(ii)]).

Lemma 2.5. *Let k be a positive integer and χ be a nontrivial primitive Dirichlet character modulo N .*

- (i) $L(1-k, \chi) \neq 0$ if and only if $\chi(-1) = (-1)^k$.
- (ii) *We have*

$$L(1-k, \chi) = -\frac{B_{k, \chi}}{k},$$

where $B_{k, \chi}$ is a generalized Bernoulli number defined by the following identity of infinite series

$$\sum_{a=1}^{N-1} \chi(a) \frac{t e^{at}}{e^{Nt} - 1} = \sum_{k=0}^{\infty} B_{k, \chi} \frac{t^k}{k!}.$$

Proof. (i) See [8, Chapter XIV Corollary of Theorem 2.2].

(ii) See [8, Chapter XIV Theorem 2.3]. \square

Proposition 2.6. *Let χ and ψ be primitive Dirichlet characters modulo L and M , respectively. Let k be an integer such that $\chi(-1)\psi(-1) = (-1)^k$. Define*

$$E_{k, \chi, \psi}(\tau) := c_0 + \sum_{n=1}^{\infty} \left(\sum_{d>0, d|n} \chi(n/d) \psi(d) d^{k-1} \right) q^n \in \mathbb{C}[[q]]$$

with

$$c_0 := \begin{cases} 0 & \text{if } L > 1 \\ L(1-k, \psi)/2 & \text{if } L = 1, \end{cases}$$

and set $\chi(d) = \psi(d) = 0$ if $\gcd(d, LM) \neq 1$. Except for the case when $k = 2$ and $\chi = \psi = 1$ the function $E_{k, \chi, \psi}(\tau)$ defines an element of $\mathcal{M}_k(LM, \chi\psi)$, which is called an *Eisenstein series*.

Proof. See [9, Theorem 4.7.1 and Lemma 7.2.19]. \square

Corollary 2.7. *Let k and N be positive integers such that $(-1)^k N$ is the discriminant of a quadratic field. Then the Eisenstein series*

$$G_{k,N}(\tau) := \frac{L(1-k, \chi_{(-1)^k N})}{2} + \sum_{n=1}^{\infty} \left(\sum_{d>0, d|n} \chi_{(-1)^k N}(d) d^{k-1} \right) q^n = \frac{L(1-k, \chi_{(-1)^k N})}{2} + q + O(q^2), \quad (2.2)$$

$$H_{k,N}(\tau) := \sum_{n=1}^{\infty} \left(\sum_{d>0, d|n} \chi_{(-1)^k N}(n/d) d^{k-1} \right) q^n = q + O(q^2) \quad (2.3)$$

are linearly independent elements of $\mathcal{M}_k(N, \chi_{(-1)^k N})$.

Proof. Set $\chi = 1$ and $\psi = \chi_{(-1)^k N}$, which are primitive Dirichlet characters modulo 1 and N (≥ 3), respectively. Since $\chi(-1)\psi(-1) = \chi_{(-1)^k N}(-1) = (-1)^k$ by the definition (2.1), it follows from Proposition 2.6 that the Eisenstein series

$$E_{k,\chi,\psi}(\tau) = \frac{L(1-k, \chi_{(-1)^k N})}{2} + \sum_{n=1}^{\infty} \left(\sum_{d>0, d|n} \chi_{(-1)^k N}(d) d^{k-1} \right) q^n = \frac{L(1-k, \chi_{(-1)^k N})}{2} + q + O(q^2)$$

belongs to $\mathcal{M}_k(N, \chi_{(-1)^k N})$. Observe that the constant term $L(1-k, \chi_{(-1)^k N})/2$ does not vanish by Lemma 2.5(i).

Similarly, if we let $\chi = \chi_{(-1)^k N}$ and $\psi = 1$, then the Eisenstein series

$$E_{k,\chi,\psi}(\tau) = \sum_{n=1}^{\infty} \left(\sum_{d>0, d|n} \chi_{(-1)^k N}(d) d^{k-1} \right) q^n = q + O(q^2)$$

belongs to $\mathcal{M}_k(N, \chi_{(-1)^k N})$ by Proposition 2.6. This completes the proof. \square

Remark 2.8. Although Corollary 2.7 was originally given by Hecke ([4, p.818]), we derive it as a direct corollary of a more generalized result (Proposition 2.6) due to Miyake.

Let k be an integer and χ be a Dirichlet character modulo N . For a positive integer m , the *Hecke operator* $\cdot|T_{m,k,\chi}$ is defined on the functions $f(\tau) = \sum_{n=0}^{\infty} a(n)q^n \in \mathcal{M}_k(N, \chi)$ by the rule

$$f(\tau)|T_{m,k,\chi} := \sum_{n=0}^{\infty} \left(\sum_{d>0, d|\gcd(m,n)} \chi(d) d^{k-1} a(mn/d^2) \right) q^n. \quad (2.4)$$

Here we set $\chi(d) = 0$ if $\gcd(N, d) \neq 1$.

Proposition 2.9. *With the notation as above, the operator $\cdot|T_{m,k,\chi}$ preserves the space $\mathcal{M}_k(N, \chi)$.*

Proof. See [7, Chapter 3 Propositions 36 and 39]. \square

3. THETA FUNCTIONS ASSOCIATED WITH QUADRATIC FORMS

Let A be an $r \times r$ positive definite symmetric matrix over \mathbb{Z} with even diagonal entries. Let Q be its associated quadratic form, namely

$$Q(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T A \mathbf{x} \quad \text{for } \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} \in \mathbb{Z}^r.$$

We define the theta function $\Theta_Q(\tau)$ on \mathbb{H} associated with Q by

$$\Theta_Q(\tau) := \sum_{\mathbf{x} \in \mathbb{Z}^r} e^{2\pi i Q(\mathbf{x})\tau} = \sum_{n=0}^{\infty} r_Q(n) q^n,$$

where

$$r_Q(n) := \#\{\mathbf{x} \in \mathbb{Z}^r ; Q(\mathbf{x}) = n\}$$

is the representation number of n by Q .

Proposition 3.1. *With the notations as above, we further assume that r is even. Let N be a positive integer such that NA^{-1} is an integral matrix with even diagonal entries. Then $\Theta_Q(\tau)$ belongs to $\mathcal{M}_{r/2}(N, \chi_{(-1)^{r/2} \det(A)})$.*

Proof. See [5, Theorem 10.9] or [9, Corollary 4.9.5]. \square

Remark 3.2. If such a matrix A exists in the statement of Proposition 3.1, then $(-1)^{r/2} \det(A) \equiv 0$ or $1 \pmod{4}$ ([5, p.180]). So $\chi_{(-1)^k \det(A)}$ makes sense.

Now we are ready to prove our main theorem as follows.

Theorem 3.3. *Let k and N be positive integers such that $(-1)^k N$ is a discriminant of a quadratic field and $\dim_{\mathbb{C}} \mathcal{M}_k(N, \chi_{(-1)^k N}) = 2$. Let A be a $2k \times 2k$ positive definite symmetric matrix over \mathbb{Z} with $\det(A) = N$ such that both A and NA^{-1} have even diagonal entries. Let Q be the quadratic form associated with A .*

(i) *We have $r_Q(0) = 1$ and*

$$r_Q(n) = c_1 \sum_{d>0, d|n} \chi_{(-1)^k N}(d) d^{k-1} + c_2 \sum_{d>0, d|n} \chi_{(-1)^k N}(n/d) d^{k-1} \quad \text{for any positive integer } n, \quad (3.1)$$

where

$$c_1 = \frac{2}{L(1-k, \chi_{(-1)^k N})} \quad \text{and} \quad c_2 = r_Q(1) - \frac{2}{L(1-k, \chi_{(-1)^k N})}. \quad (3.2)$$

(ii) *Let p be a prime not dividing N . If m is a nonnegative integer such that $\chi_{(-1)^k N}(p^m) = 1$ (this condition holds true whenever m is even), then we have the identity*

$$r_Q(1)r_Q(p^m n) = r_Q(p^m)r_Q(n) \quad \text{for any positive integer } n \text{ prime to } p.$$

Proof. (i) Since A is positive definite, $r_Q(0) = 1$. Consider the theta function $\Theta_Q(\tau) = \sum_{n=0}^{\infty} r_Q(n) q^n$. Then it lies in $\mathcal{M}_k(N, \chi_{(-1)^k N})$ by Proposition 3.1. Since we are assuming that $\dim_{\mathbb{C}} \mathcal{M}_k(N, \chi_{(-1)^k N}) = 2$, we see from Corollary 2.7 that

$$\Theta_Q(\tau) = c_1 G_{k,N}(\tau) + c_2 H_{k,N}(\tau) \quad \text{for some } c_1, c_2 \in \mathbb{C}. \quad (3.3)$$

We can then determine c_1 and c_2 in (3.2) by observing the first two terms of

$$\Theta_Q(\tau) = 1 + r_Q(1)q + O(q^2), \quad G_{k,N}(\tau) = \frac{L(1-k, \chi_{(-1)^k N})}{2} + q + O(q^2) \quad \text{and} \quad H_{k,N}(\tau) = q + O(q^2).$$

By (3.3) and the definitions (2.2), (2.3) we obtain a general formula (3.1) for $r_Q(n)$ ($n \geq 1$).

(ii) Let p be a prime not dividing N and m be a nonnegative integer such that $\chi_{(-1)^k N}(p^m) = 1$. By the formula (3.1) we get that

$$\begin{aligned} r_Q(p^m) &= c_1 \sum_{a=0}^m \chi_{(-1)^k N}(p^a) p^{a(k-1)} + c_2 \sum_{a=0}^m \chi_{(-1)^k N}(p^{m-a}) p^{a(k-1)} \\ &= (c_1 + c_2) \sum_{a=0}^m \chi_{(-1)^k N}(p^a) p^{a(k-1)} \quad \text{by the facts } \chi_{(-1)^k N}(p^m) = 1 \text{ and } \chi_{(-1)^k N}(p) = \pm 1 \\ &= r_Q(1) \sum_{a=0}^m \chi_{(-1)^k N}(p^a) p^{a(k-1)} \quad \text{by (3.2)}. \end{aligned} \quad (3.4)$$

On the other hand, we deduce that

$$\begin{aligned} r_Q(1)\Theta_Q(\tau)|_{T_{p^m, k, \chi_{(-1)^k N}}} &= r_Q(1) \left(r_Q(0) \sum_{a=0}^m \chi_{(-1)^k N}(p^a) p^{a(k-1)} + r_Q(p^m)q + O(q^2) \right) \quad \text{by the definition (2.4)} \\ &= r_Q(0)r_Q(p^m) + r_Q(1)r_Q(p^m)q + O(q^2) \quad \text{by (3.4)} \\ &= r_Q(p^m)(r_Q(0) + r_Q(1)q + O(q^2)), \end{aligned}$$

which turns out to be an element of $\mathcal{M}_k(N, \chi_{(-1)^k N})$ by Proposition 2.9. Taking the set $\{\Theta_Q(\tau) = 1 + r_Q(1)q + O(q^2), H_{k,N}(\tau) = q + O(q^2)\}$ as a basis of the space $\mathcal{M}_k(N, \chi_{(-1)^k N})$ one can derive

$$r_Q(1)\Theta_Q(\tau)|T_{p^m, k, \chi_{(-1)^k N}} = r_Q(p^m)\Theta_Q(\tau).$$

Therefore, comparing the Fourier coefficients of the term q^n for any positive integer n prime to p we achieve the identity

$$r_Q(1)r_Q(p^m n) = r_Q(p^m)r_Q(n)$$

as desired. \square

Remark 3.4. (i) Let p be a prime and m be a nonnegative integer. If p divides N , then $\chi_{(-1)^k N}(p) = 0$ and we get from (3.1) that

$$r_Q(p^m) = c_1 + c_2 p^{m(k-1)}. \quad (3.5)$$

If p does not divide N , then we see from (3.1) that

$$\begin{aligned} r_Q(p^m) &= c_1 \sum_{a=0}^m \chi_{(-1)^k N}(p^a) p^{a(k-1)} + c_2 \sum_{a=0}^m \chi_{(-1)^k N}(p^{m-a}) p^{a(k-1)} \\ &= (c_1 + c_2 \chi_{(-1)^k N}(p)^m) \sum_{a=0}^m (\chi_{(-1)^k N}(p) p^{k-1})^a \quad \text{by the fact } \chi_{(-1)^k N}(p)^2 = 1 \\ &= (c_1 + c_2 \chi_{(-1)^k N}(p)^m) \frac{1 - (\chi_{(-1)^k N}(p) p^{k-1})^{m+1}}{1 - \chi_{(-1)^k N}(p) p^{k-1}}. \end{aligned} \quad (3.6)$$

(ii) Assume that N is a prime. Let n (≥ 2) be an integer with prime factorization

$$n = N^m \prod_{i=1}^t p_i^{m_i} \quad (m, m_i \geq 0).$$

If $r_Q(1) \neq 0$, then we have by Theorem 3.3(ii)

$$r_Q(n^2) = r_Q(N^{2m} \prod_{i=1}^t p_i^{2m_i}) = \frac{r_Q(N^{2m} \prod_{i=1}^{t-1} p_i^{2m_i}) r_Q(p_t^{2m_t})}{r_Q(1)} = \dots = \frac{r_Q(N^{2m}) \prod_{i=1}^t r_Q(p_i^{2m_i})}{r_Q(1)^t}.$$

Therefore by (3.5) and (3.6) one can get a concise formula for $r_Q(n^2)$.

Let $(k, N) \in \{(2, 5), (2, 13), (2, 17), (3, 3), (4, 5), (5, 3)\}$ as in Corollary 2.4. Then for each pair (k, N) one can find some matrices to which Theorem 3.3 and Remark 3.4 can be applied. However, it doesn't seem to be known how to find such matrices systematically. We close this section by giving a table for these examples.

TABLE 1. Theta functions associated with quadratic forms

(k, N)	(2, 5)	(2, 13)	(2, 17)	(3, 3)	(4, 5)	(5, 3)
$2k \times 2k$ symmetric matrix A with $\det(A) = N$	$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$
eigenvalues of A	1, 1, 1, 5	$\frac{5}{2} \pm \frac{\sqrt{9+4\sqrt{3}}}{2},$ $\frac{5}{2} \pm \frac{\sqrt{9-4\sqrt{3}}}{2}$	$\frac{5+\sqrt{2}}{2} \pm \frac{\sqrt{7+2\sqrt{2}}}{2},$ $\frac{5-\sqrt{2}}{2} \pm \frac{\sqrt{7-2\sqrt{2}}}{2}$	$1, 3, 2 \pm \frac{\sqrt{6}}{2} \pm \frac{\sqrt{2}}{2}$	eight (distinct) positive zeros of $x^8 - 20x^7 + 162x^6 - 684x^5$ $+1611x^4 - 2092x^3 + 1370x^2$ $-352x + 5$	1, 3, eight (distinct) positive zeros of $x^8 - 18x^7 + 130x^6 - 486x^5 + 1007x^4$ $-1142x^3 + 646x^2 - 140x + 1$
diagonal entries of NA^{-1}	4, 4, 4, 4	14, 4, 10, 12	12, 14, 10, 6	6, 4, 4, 10, 10, 18	12, 38, 78, 132, 50, 28, 12, 2	2, 2, 12, 42, 90, 156, 60, 36, 18, 6
quadratic form Q associated with A	$x_1^2 + x_2^2 + x_3^2 + x_4^2$ $+x_1x_2 + x_1x_3 + x_1x_4$ $+x_2x_3 + x_2x_4 + x_3x_4$	$x_1^2 + 2x_2^2 + x_3^2 + x_4^2$ $+x_1x_3 + x_1x_4 + x_2x_4$	$x_1^2 + x_2^2 + x_3^2 + 2x_4^2$ $+x_1x_2 + x_2x_4 + x_3x_4$	$x_1^2 + x_2^2 + x_3^2 + x_4^2$ $+x_5^2 + x_6^2 + x_1x_6 + x_2x_5$ $+x_3x_4 + x_4x_6 + x_5x_6$	$x_1^2 + x_2^2 + x_3^2 + x_4^2$ $+2x_5^2 + x_6^2 + x_7^2 + 2x_8^2$ $+x_1x_2 + x_2x_3 + x_3x_4$ $+2x_4x_5 + x_5x_6 + x_6x_7 + x_7x_8$	$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + 2x_7^2$ $+x_8^2 + x_9^2 + x_{10}^2 + x_1x_2 + x_3x_4 + x_4x_5$ $+x_5x_6 + 2x_6x_7 + x_7x_8 + x_8x_9 + x_9x_{10}$
$r_Q(1)$	20	12	8	72	126	246
$(1-k, \chi_{(-1)^k N})$	$-\frac{2}{5}$	-2	-4	$-\frac{2}{9}$	2	$\frac{2}{3}$
$\Theta_Q(\tau)$	$-5G_{2,5}(\tau) + 25H_{2,5}(\tau)$ $= 1 + 20q + 30q^2 + \dots$	$-G_{2,13}(\tau) + 13H_{2,13}(\tau)$ $= 1 + 12q + 14q^2 + \dots$	$-\frac{1}{2}G_{2,17}(\tau) + \frac{17}{2}H_{2,17}(\tau)$ $= 1 + 8q + 24q^2 + 18q^3 + \dots$	$-9G_{3,3}(\tau) + 81H_{3,3}(\tau)$ $= 1 + 72q + 270q^2 + \dots$	$G_{4,5}(\tau) + 125H_{4,5}(\tau)$ $= 1 + 126q + 868q^2 + \dots$	$3G_{5,3}(\tau) + 243H_{5,3}(\tau)$ $= 1 + 246q + 3600q^2 + \dots$

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